

SCHATTEN p -NORM INEQUALITIES RELATED TO AN EXTENDED OPERATOR PARALLELOGRAM LAW

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ABSTRACT. Let \mathcal{C}_p be the Schatten p -class for $p > 0$. Generalizations of the parallelogram law for the Schatten 2-norms have been given in the following form: If $\mathbf{A} = \{A_1, A_2, \dots, A_n\}$ and $\mathbf{B} = \{B_1, B_2, \dots, B_n\}$ are two sets of operators in \mathcal{C}_2 , then

$$\sum_{i,j=1}^n \|A_i - A_j\|_2^2 + \sum_{i,j=1}^n \|B_i - B_j\|_2^2 = 2 \sum_{i,j=1}^n \|A_i - B_j\|_2^2 - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_2^2.$$

In this paper, we give generalizations of this as pairs of inequalities for Schatten p -norms, which hold for certain values of p and reduce to the equality above for $p = 2$. Moreover, we present some related inequalities for three sets of operators.

1. INTRODUCTION

Suppose that $\mathbb{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on a separable complex Hilbert space \mathcal{H} endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $A \in \mathbb{B}(\mathcal{H})$ be a compact operator and let $\{s_j(A)\}$ denote the sequence of decreasingly ordered singular values of A , i.e. the eigenvalues of $|A| = (A^*A)^{1/2}$. The Schatten p -norm (p -quasi-norm, resp.) for $1 \leq p < \infty$ ($0 < p < 1$, resp.) is defined by

$$\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p}.$$

For $p > 0$, the Schatten p -class, denoted by \mathcal{C}_p , is defined to be the two-sided ideal in $\mathbb{B}(\mathcal{H})$ of those compact operators A for which $\|A\|_p$ is finite. Clearly

$$(1.1) \quad \left\| |A|^2 \right\|_{p/2} = \|A\|_p^2$$

for $p > 0$. In particular, \mathcal{C}_1 and \mathcal{C}_2 are the trace class and the Hilbert-Schmidt class, respectively. For $1 \leq p < \infty$, \mathcal{C}_p is a Banach space; in particular the triangle inequality holds. For $0 < p < 1$, the quasi-norm $\|\cdot\|_p$ does not satisfy the triangle inequality, but

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however satisfies the inequality $\|A + B\|_p^p \leq \|A\|_p^p + \|B\|_p^p$. For more information on the theory of Schatten p -norms the reader is referred to [8, Chapter 2].

It follows from [7, Corollary 2.7] that for $A_1, \dots, A_n, B_1, \dots, B_n \in \mathbb{B}(\mathcal{H})$

$$(1.2) \quad \sum_{i,j=1}^n \|A_i - A_j\|^2 + \sum_{i,j=1}^n \|B_i - B_j\|^2 = 2 \sum_{i,j=1}^n \|A_i - B_j\|^2 - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|^2,$$

which is indeed a generalization of the classical *parallelogram law*:

$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2 \quad (z, w \in \mathbb{C}).$$

There are several extensions of parallelogram law among them we could refer the interested reader to [2, 3, 6, 9, 10]. Generalizations of the parallelogram law for the Schatten p -norms have been given in the form of the celebrated Clarkson inequalities (see [4] and references therein). Since \mathcal{C}_2 is a Hilbert space under the inner product $\langle A, B \rangle = \text{tr}(B^* A)$, it follows from an equality similar to (1.2) stated for vectors of a Hilbert space (see [7, Corollary 2.7]) that if $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{C}_2$, then

$$(1.3) \quad \sum_{i,j=1}^n \|A_i - A_j\|_2^2 + \sum_{i,j=1}^n \|B_i - B_j\|_2^2 = 2 \sum_{i,j=1}^n \|A_i - B_j\|_2^2 - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_2^2.$$

In [7] a joint operator extension of the Bohr and parallelogram inequalities is presented. In particular, it follows from [7, Corollary 2.3] that if $A_1, \dots, A_n, B_1, \dots, B_n \in \mathbb{B}(\mathcal{H})$, then

$$(1.4) \quad \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 + \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 = \sum_{i,j=1}^n |A_i - B_j|^2 - \left| \sum_{i=1}^n (A_i - B_i) \right|^2.$$

In this paper, we give a generalization of the equality (1.3) for the Schatten p -norms ($p > 0$). First we present similar consideration for three sets of operators. In addition, we provide pairs of complementary inequalities that reduce to (1.3) for the certain value $p = 2$.

2. SCHATTEN p -NORM INEQUALITIES

To accomplish our results, we need the following lemma which can be deduced from [1, Lemma 4] and [8, p. 20]:

Lemma A. Let $A_1, \dots, A_n \in \mathcal{C}_p$ for some $p > 0$. If A_1, \dots, A_n are positive, then

$$(2.1) \quad n^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p \leq \sum_{i=1}^n \|A_i\|_p^p$$

for $0 < p \leq 1$ and the reverse inequalities hold for $1 \leq p < \infty$.

Let us define a constant $D_{\mathbf{A}}$ for a set of operators $\mathbf{A} = \{A_1, A_2, \dots, A_n\}$ as follows:

$$D_{\mathbf{A}} := \sum_{i=1}^n \delta(A_i) \quad \text{where} \quad \delta(A_i) = \begin{cases} 1 & (A_i \neq 0) \\ 0 & (A_i = 0) \end{cases}.$$

If there exists $1 \leq i \leq n$ with $A_i = 0$, then Lemma A is refined as follows:

$$(2.2) \quad D_{\mathbf{A}}^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p \leq \sum_{i=1}^n \|A_i\|_p^p$$

for $0 < p \leq 1$ and the reverse inequalities hold for $1 \leq p < \infty$.

We also put $\mathbf{A} - \mathbf{B} := \{A_i - B_j : 1 \leq i, j \leq n\}$ for sets of operators $\mathbf{A} = \{A_1, A_2, \dots, A_n\}$ and $\mathbf{B} = \{B_1, B_2, \dots, B_n\}$. Then we remark that $0 \leq D_{\mathbf{A}-\mathbf{B}} \leq n^2$.

Now we give our main results that involve three sets of operators.

Theorem 2.1. *Let $\mathbf{A} = \{A_1, A_2, \dots, A_n\}$, $\mathbf{B} = \{B_1, B_2, \dots, B_n\}$, $\mathbf{C} = \{C_1, C_2, \dots, C_n\} \subset \mathcal{C}_p$ for some $p > 0$. Then*

$$(2.3) \quad \begin{aligned} & \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p + \sum_{i,j=1}^n \|C_i - C_j\|_p^p \\ & \geq \left(D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p + D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|B_i - C_j\|_p^p + D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|C_i - A_j\|_p^p \right) \\ & \quad - \left(\left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^n (B_i - C_i) \right\|_p^p + \left\| \sum_{i=1}^n (C_i - A_i) \right\|_p^p \right) \end{aligned}$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only prove the case when $0 < p \leq 2$. The other case can be proved by a similar argument.

We have

$$\begin{aligned} & \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p + \sum_{i,j=1}^n \|C_i - C_j\|_p^p \\ & + \left(\left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^n (B_i - C_i) \right\|_p^p + \left\| \sum_{i=1}^n (C_i - A_i) \right\|_p^p \right) \\ & = 2 \left(\sum_{1 \leq i < j \leq n} \|A_i - A_j\|_p^p + \sum_{1 \leq i < j \leq n} \|B_i - B_j\|_p^p + \sum_{1 \leq i < j \leq n} \|C_i - C_j\|_p^p \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^n (B_i - C_i) \right\|_p^p + \left\| \sum_{i=1}^n (C_i - A_i) \right\|_p^p \right) \\
& = 2 \left(\sum_{1 \leq i < j \leq n} \| |A_i - A_j|^2 \|_{p/2}^{p/2} + \sum_{1 \leq i < j \leq n} \| |B_i - B_j|^2 \|_{p/2}^{p/2} + \sum_{1 \leq i < j \leq n} \| |C_i - C_j|^2 \|_{p/2}^{p/2} \right) \\
& + \left(\left\| \sum_{i=1}^n (A_i - B_i) \right\|_{p/2}^{2p/2} + \left\| \sum_{i=1}^n (B_i - C_i) \right\|_{p/2}^{2p/2} + \left\| \sum_{i=1}^n (C_i - A_i) \right\|_{p/2}^{2p/2} \right) \\
& \quad \text{(by relation (1.1))} \\
& \geq \left\| \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 + \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 + \sum_{i=1}^n (A_i - B_i) \right\|_{p/2}^{2p/2} \\
& + \left\| \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 + \sum_{1 \leq i < j \leq n} |C_i - C_j|^2 + \sum_{i=1}^n (B_i - C_i) \right\|_{p/2}^{2p/2} \\
& + \left\| \sum_{1 \leq i < j \leq n} |C_i - C_j|^2 + \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 + \sum_{i=1}^n (C_i - A_i) \right\|_{p/2}^{2p/2} \\
& \quad \text{(by the second inequality of (2.1))} \\
& = \left\| \sum_{i,j=1}^n |A_i - B_j|^2 \right\|_{p/2}^{p/2} + \left\| \sum_{i,j=1}^n |B_i - C_j|^2 \right\|_{p/2}^{p/2} + \left\| \sum_{i,j=1}^n |C_i - A_j|^2 \right\|_{p/2}^{p/2} \\
& \quad \text{(by (1.4))} \\
& \geq D_{\mathbf{A}-\mathbf{B}}^{\frac{p}{2}-1} \sum_{i,j=1}^n \| |A_i - B_j|^2 \|_{p/2}^{p/2} + D_{\mathbf{B}-\mathbf{C}}^{\frac{p}{2}-1} \sum_{i,j=1}^n \| |B_i - C_j|^2 \|_{p/2}^{p/2} + D_{\mathbf{C}-\mathbf{A}}^{\frac{p}{2}-1} \sum_{i,j=1}^n \| |C_i - A_j|^2 \|_{p/2}^{p/2} \\
& \quad \text{(by the first inequality of (2.2))} \\
& = D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \| A_i - B_j \|_p^p + D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \| B_i - C_j \|_p^p + D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \| C_i - A_j \|_p^p.
\end{aligned}$$

So we have the desired inequality (2.3). □

The next result can be regarded as a generalization of (1.3).

Proposition 2.2. *Let $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{C}_p$ for some $p > 0$. Then*

$$\begin{aligned} & \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p \\ & \geq 2n^{p-2} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \end{aligned}$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only prove the case where $0 < p \leq 2$. Putting $C_i = 0$ for $i = 1, 2, \dots, n$, we have

$$D_{\mathbf{B}-\mathbf{C}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|B_i - C_j\|_p^p = n D_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^n \|B_i\|_p^p, \quad D_{\mathbf{C}-\mathbf{A}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|C_i - A_j\|_p^p = n D_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^n \|A_i\|_p^p.$$

It follows from Theorem 2.1 that

$$\begin{aligned} 0 & \leq \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p - D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p \\ & \quad - n \left(D_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^n \|B_i\|_p^p + D_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^n \|A_i\|_p^p \right) \\ & \quad + \left(\left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p + \left\| \sum_{i=1}^n B_i \right\|_p^p + \left\| \sum_{i=1}^n A_i \right\|_p^p \right) \\ & = \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p - 2 D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p + 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \\ & \quad + \left(D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \right) \\ & \quad + \left(\left\| \sum_{i=1}^n A_i \right\|_p^p - n D_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^n \|A_i\|_p^p \right) + \left(\left\| \sum_{i=1}^n B_i \right\|_p^p - n D_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^n \|B_i\|_p^p \right). \end{aligned}$$

Here the inequality (2.2) implies

$$\begin{aligned} & D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \\ & = D_{\mathbf{A}-\mathbf{B}}^{\frac{p}{2}-1} \sum_{i,j=1}^n \| |A_i - B_j|^2 \|_{p/2}^{p/2} - \left\| \sum_{i=1}^n |A_i - B_i|^2 \right\|_{p/2}^{p/2} \leq 0 \end{aligned}$$

by relation (1.1). Due to $nD_{\mathbf{A}}^{\frac{p-2}{2}}$ is no less than 1, we deduce from Lemma A that

$$\left\| \sum_{i=1}^n A_i \right\|_p^p - nD_{\mathbf{A}}^{\frac{p-2}{2}} \sum_{i=1}^n \|A_i\|_p^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p - \sum_{i=1}^n \|A_i\|_p^p \leq 0.$$

Similarly, we have $\|\sum_{i=1}^n B_i\|_p^p \leq nD_{\mathbf{B}}^{\frac{p-2}{2}} \sum_{i=1}^n \|B_i\|_p^p$. It therefore implies that

$$\begin{aligned} \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p &\geq 2D_{\mathbf{A}-\mathbf{B}}^{\frac{p-2}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \\ &\geq 2n^{p-2} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \geq 0 \\ &\quad (\text{by } n^2 \geq D_{\mathbf{A}-\mathbf{B}}(\geq 0)). \end{aligned}$$

Thus we obtain the desired inequality. \square

Corollary 2.3. *Let $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{C}_p$ for some $p > 0$ and $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i$. Then*

$$\sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p \geq 2n^{p-2} \sum_{i,j=1}^n \|A_i - B_j\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Utilizing Corollary 2.3 with $B_i = 0$ ($1 \leq i \leq n$), we obtain the following result which is a refinement of [5, Corollary 2.3].

Corollary 2.4. *Let $A_1, \dots, A_n \in \mathcal{C}_p$ for some $p > 0$ such that $\sum_{i=1}^n A_i = 0$. Then*

$$\sum_{i,j=1}^n \|A_i - A_j\|_p^p \geq 2n^{p-1} \sum_{i=1}^n \|A_i\|_p^p$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Next, we have the following reverse inequalities of Proposition 2.2:

Proposition 2.5. *Let $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{C}_p$ for some $p > 0$. Then*

$$\begin{aligned} & 2(n^2 - n + 1)^{\frac{2-p}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p - 2 \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \\ & \geq \sum_{i,j=1}^n \|A_i - A_j\|_p^p + \sum_{i,j=1}^n \|B_i - B_j\|_p^p \end{aligned}$$

for $0 < p \leq 2$ and the reverse inequality holds for $2 \leq p < \infty$.

Proof. We only consider the case when $0 < p \leq 2$. We have

$$\begin{aligned} & \sum_{1 \leq i < j \leq n} \|A_i - A_j\|_p^p + \sum_{1 \leq i < j \leq n} \|B_i - B_j\|_p^p + \left\| \sum_{i=1}^n (A_i - B_i) \right\|_p^p \\ = & \sum_{1 \leq i < j \leq n} \| |A_i - A_j|^2 \|_{p/2}^{p/2} + \sum_{1 \leq i < j \leq n} \| |B_i - B_j|^2 \|_{p/2}^{p/2} + \left\| \sum_{i=1}^n (A_i - B_i) \right\|_{p/2}^{2p/2} \\ & \quad \text{(by relation (1.1))} \\ \leq & \left(2 \cdot \frac{n^2 - n}{2} + 1 \right)^{1 - \frac{p}{2}} \left\| \sum_{1 \leq i < j \leq n} |A_i - A_j|^2 + \sum_{1 \leq i < j \leq n} |B_i - B_j|^2 + \left| \sum_{i=1}^n (A_i - B_i) \right|^2 \right\|_{p/2}^{p/2} \\ & \quad \text{(by the first inequality of (2.1))} \\ = & (n^2 - n + 1)^{\frac{2-p}{2}} \left\| \sum_{i,j=1}^n |A_i - B_j|^2 \right\|_{p/2}^{p/2} \quad \text{(by (1.4))} \\ \leq & (n^2 - n + 1)^{\frac{2-p}{2}} \sum_{i,j=1}^n \| |A_i - B_j|^2 \|_{p/2}^{p/2} \quad \text{(by the second inequality of (2.1))} \\ = & (n^2 - n + 1)^{\frac{2-p}{2}} \sum_{i,j=1}^n \|A_i - B_j\|_p^p. \end{aligned}$$

So we have the desired inequality. \square

Remark 2.6. (i) By an easily calculation, we have the inequality $n^{p-2} < (2n^2 - n + 1)^{\frac{2-p}{2}} < (n^{p-2})^{-1}$ for $0 < p \leq 2$.

(ii) The values of $2n^{p-2}$, $2(n^2 - n + 1)^{\frac{2-p}{2}}$ of Propositions 2.2 and 2.5 are 2, if $p = 2$. So these results ensured the equality (1.3).

Finally we would like to give a problem for further research.

Problem 2.7. What the form of the identity is for the general case of k sets of operators?

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